Recursive Versions of the Reassigned Spectrogram and of the Synchrosqueezed STFT

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I) Introduction

Goals of this study

Computing time-frequency representations that are

- candidate for a real-time implementation
- adjustable by the user through a damping parameter μ (Levenberg-Marquardt approach)
- able to extract modes and to reconstruct the signal (synchrosqueezing)
- based on a filter bank approach

Proposed approach

- use of a particular case of the STFT with a causal, infinite length window function that can be rewritten as a causal IIR recursive filtering
- the algorithmic complexity depends on the filter order and on the analyzed frequency bandwidth

II) Filter-based reassigned and synchrosqueezed STFT

The STFT as a convolution product

The STFT of a signal x using a real-valued analysis window h, denoted $F_x^h(t,\omega)=M_x^h(t,\omega)\,\mathbf{e}^{j\Phi_x^h(t,\omega)}$ can be related to the linear convolution product between the analyzed signal x and the complex valued impulse response of a bandpass filter $g(t,\omega)=h(t)\,\mathbf{e}^{j\omega t}$:

$$y_x^g(t,\omega)=\int_{-\infty}^{+\infty} g(\tau,\omega)x(t-\tau)\,d\tau=|y_x^g(t,\omega)|\,\mathbf{e}^{j\Psi_x^g(t,\omega)}$$
$$=F_x^h(t,\omega)\,\mathbf{e}^{j\omega t}=M_x^h(t,\omega)\,\mathbf{e}^{j(\Phi_x^h(t,\omega)+\omega t)}$$

thus

$$M_x^h(t,\omega)=|y_x^g(t,\omega)| \text{ and } \Phi_x^h(t,\omega)=\Psi_x^g(t,\omega)-\omega t.$$

Rewording the reassignment operators of the spectrogram

According to [1], the spectrogram reassignment operators can be reformulated using the phase of $y_x^g(t,\omega)$, denoted $\Psi_x^g(t,\omega)=\Phi_x^h(t,\omega)+\omega t$

$$\hat{t}(t,\omega)=-\frac{\partial\Phi_x^h}{\partial\omega}(t,\omega)=t-\frac{\partial\Psi_x^g}{\partial\omega}(t,\omega), \quad (1)$$

$$\hat{\omega}(t,\omega)=\omega+\frac{\partial\Phi_x^h}{\partial t}(t,\omega)=\frac{\partial\Psi_x^g}{\partial t}(t,\omega). \quad (2)$$

The reassigned spectrogram is expressed as

$$\mathrm{RSP}(t,\omega)=\iint_{\mathbb{R}^2}|y_x^g(t',\omega')|^2\delta(t-\hat{t}(t',\omega'))\,\delta(\omega-\hat{\omega}(t',\omega'))\,dt'd\omega' \quad (3)$$

where $\delta(t)$ denotes the Dirac distribution.

Rewording the Levenberg-Marquardt reassignment [2]

$$\begin{pmatrix}\tilde{t}(t,\omega)\\ \tilde{\omega}(t,\omega)\end{pmatrix}=\begin{pmatrix}t\\ \omega\end{pmatrix}-\left(\nabla^t R_x^h(t,\omega)+\mu I_2\right)^{-1}R_x^h(t,\omega) \quad (4)$$

$$R_x^h(t,\omega)=\begin{pmatrix}t-\hat{t}(t,\omega)\\ \omega-\hat{\omega}(t,\omega)\end{pmatrix}=\begin{pmatrix}\frac{\partial\Psi_x^g}{\partial\omega}(t,\omega)\\ \omega-\frac{\partial\Psi_x^g}{\partial t}(t,\omega)\end{pmatrix}$$

$$\nabla^t R_x^h(t,\omega)=\begin{pmatrix}\frac{\partial^2\Psi_x^g}{\partial t\partial\omega}(t,\omega) & \frac{\partial^2\Psi_x^g}{\partial\omega^2}(t,\omega)\\ -\frac{\partial^2\Psi_x^g}{\partial t^2}(t,\omega) & 1-\frac{\partial^2\Psi_x^g}{\partial t\partial\omega}(t,\omega)\end{pmatrix}$$

Hence, the Levenberg-Marquardt reassigned spectrogram ($\mathrm{LMRSP}(t,\omega)$) is obtained by replacing $(\hat{t},\hat{\omega})$ by $(\tilde{t},\tilde{\omega})$ in Eq. (3).

Rewording the partial derivatives of the phase

$$\frac{\partial\Psi_x^g}{\partial t}(t,\omega)=\mathrm{Im}\left(\frac{y_x^{\mathcal{D}g}(t,\omega)}{y_x^g(t,\omega)}\right)$$

$$\frac{\partial\Psi_x^g}{\partial\omega}(t,\omega)=\mathrm{Re}\left(\frac{y_x^{\mathcal{T}g}(t,\omega)}{y_x^g(t,\omega)}\right)$$

$$\frac{\partial^2\Psi_x^g}{\partial t\partial\omega}(t,\omega)=\mathrm{Re}\left(\frac{y_x^{\mathcal{DT}g}(t,\omega)}{y_x^g(t,\omega)}-\frac{y_x^{\mathcal{D}g}(t,\omega)y_x^{\mathcal{T}g}(t,\omega)}{y_x^g(t,\omega)^2}\right)$$

$$\frac{\partial^2\Psi_x^g}{\partial t^2}(t,\omega)=\mathrm{Im}\left(\frac{y_x^{\mathcal{D}^2g}(t,\omega)}{y_x^g(t,\omega)}-\left(\frac{y_x^{\mathcal{D}g}(t,\omega)}{y_x^g(t,\omega)}\right)^2\right)$$

$$\frac{\partial^2\Psi_x^g}{\partial\omega^2}(t,\omega)=-\mathrm{Im}\left(\frac{y_x^{\mathcal{T}^2g}(t,\omega)}{y_x^g(t,\omega)}-\left(\frac{y_x^{\mathcal{T}g}(t,\omega)}{y_x^g(t,\omega)}\right)^2\right)$$

where y_x^g, $y_x^{\mathcal{T}g}$, $y_x^{\mathcal{D}g}$, $y_x^{\mathcal{DT}g}$, $y_x^{\mathcal{T}^2g}$ and $y_x^{\mathcal{D}^2g}$ are the outputs of the filters using respectively the impulse responses $g(t,\omega)$, $\mathcal{T}g=t\,g(t,\omega)$, $\mathcal{D}g(t,\omega)=\frac{\partial g}{\partial t}(t,\omega)$, $\mathcal{DT}g(t,\omega)=\frac{\partial}{\partial t}(t\,g(t,\omega))$, $\mathcal{T}^2g(t,\omega)=t^2g(t,\omega)$ and $\mathcal{D}^2g(t,\omega)=\frac{\partial^2 g}{\partial t^2}(t,\omega)$.

Rewording the synchrosqueezed STFT

y_x^g admits the following signal reconstruction formula

$$x(t-t_0)=\frac{1}{h(t_0)}\int_{-\infty}^{+\infty}y_x^g(t,\omega)\,\mathbf{e}^{-j\omega t_0}\frac{d\omega}{2\pi},\ \text{when } h(t_0)\neq 0. \quad (5)$$

This leads to the synchrosqueezed STFT [5] :

$$\mathrm{Sy}_x^g(t,\omega)=\int_{\mathbb{R}}y_x^g(t,\omega')\,\mathbf{e}^{-j\omega' t_0}\delta\left(\omega-\hat{\omega}(t,\omega')\right)\,d\omega' \quad (6)$$

$\mathrm{LMSy}_x^g(t,\omega)$ is obtained by replacing $\hat{\omega}$ by $\tilde{\omega}$. A sharpen time-frequency representation is provided by $|\mathrm{Sy}_x^g(t,\omega)|^2$. The signal can be reconstructed from $\mathrm{Sy}_x^g(t,\omega)$ as

$$\hat{x}(t-t_0)=\frac{1}{h(t_0)}\int_{-\infty}^{+\infty}\mathrm{Sy}_x^g(t,\omega)\,\frac{d\omega}{2\pi} \quad (7)$$

III) Towards a recursive implementation

Recursive implementation [3]

y_x^g can be recursively implemented using

$$h_k(t)=\frac{t^{k-1}}{T^k(k-1)!}\,\mathbf{e}^{-t/T}\,U(t), \quad (8)$$

$$g_k(t,\omega)=h_k(t)\,\mathbf{e}^{j\omega t}=\frac{t^{k-1}}{T^k(k-1)!}\,\mathbf{e}^{pt}\,U(t) \quad (9)$$

with $p=-\frac{1}{T}+j\omega$, $k\geq 1$ being the filter order, T the time spread of the window and $U(t)$ the Heaviside step function. **The time derivatives of $g_k(t,\omega)$ or its products by t and t^2 can be expressed as a linear combination of $g_k(t,\omega)$ with different filter orders.**

Discretization using the impulse invariance method [4]

$$G_k(z,\omega)=T_s\mathcal{Z}\left\{g_k(t,\omega)\right\}=\frac{\sum_{i=0}^{k-1}b_iz^{-i}}{1+\sum_{i=1}^{k}a_iz^{-i}} \quad (10)$$

with $b_i=(L^k(k-1)!)^{-1}B_{k-1,k-i-1}\alpha^i$, $\alpha=\mathbf{e}^{pT_s}$, $L=T/T_s$, $a_i=A_{k,i}\,(-\alpha)^i$, T_s being the sampling period. $B_{k,i}=\sum_{j=0}^{i}(-1)^jA_{k+1,j}(i+1-j)^k$ denotes the Eulerian numbers and $A_{k,i}$ the binomial coefficients. Hence, using $y_k[n,m]\approx y_x^{g_k}(nT_s,\frac{2\pi m}{MT_s})$ with $n\in\mathbb{Z}$ and $m=0,1,\ldots,M-1$, we obtain

$$y_k[n,m]=\sum_{i=0}^{k-1}b_i\,x[n-i]-\sum_{j=1}^{k}a_j\,y_k[n-j,m] \quad (11)$$

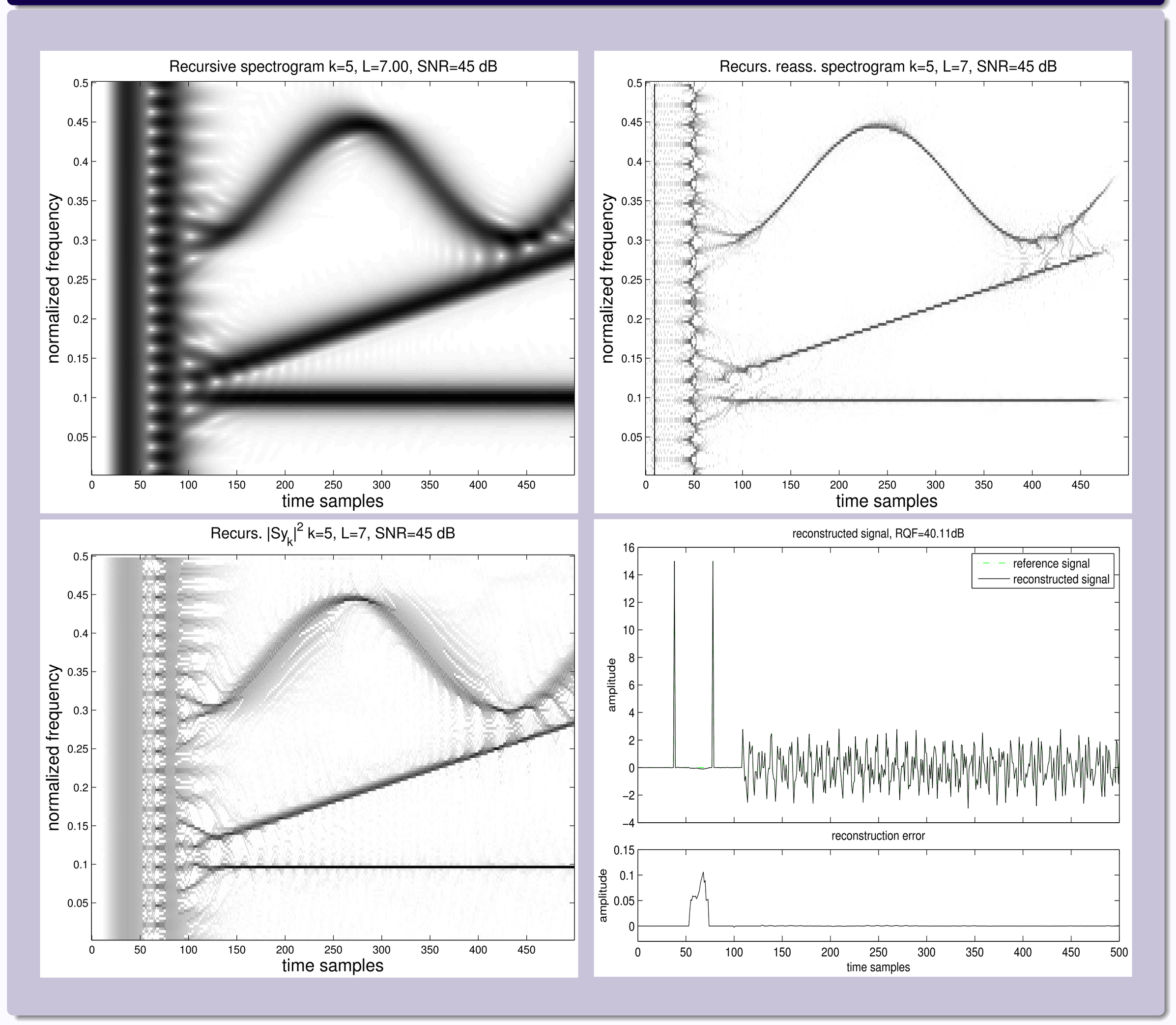
Algorithm implementation for recursive TFR computation

At time index n :

1. Compute the required $y_k^g[n,m]$ using $x[n-i]$ and $y_k[n-j,m]$ with $i\in[0,k-1]$, $j\in[1,k]$
2. Compute the other required specific filtered signals (*i.e.* $y_k^{\mathcal{T}g}$, $y_k^{\mathcal{D}g}$, $y_k^{\mathcal{DT}g}$, $y_k^{\mathcal{T}^2g}$ or $y_k^{\mathcal{D}^2g}$) using y_k^g with different filter orders
3. Compute the reassignment operators $\hat{n}$,$\hat{m}$ (resp. $\tilde{n}$, $\tilde{m}$)
4. **If** $\hat{n}\leq n$ (resp. $\tilde{n}\leq n$) **then** update $\mathrm{TFR}[\hat{n},m]$ **else** store the triplet $\left(y_k^g[n,m],\hat{n},m\right)$ into a list
5. Update $\mathrm{TFR}[n,m]$ using all previously stored triplets such as $\hat{n}=n$ (resp. $\tilde{n}=n$) and remove them from the list

IV) Numerical results

Resulting time-frequency representations



Signal reconstruction quality

Using the discrete-time version of Eq. (7) with $n_0=t_0/T_s$.

(a)	n_0	8	18	26	**28**	30
	RQF (dB)	9.79	24.17	26.77	**26.82**	26.73
(b)	M	100	200	600	1000	**2400**
	RQF (dB)	20.56	24.90	29.48	30.50	**30.87**
(c)	μ	0.30	0.80	1.30	1.80	**2.30**
	RQF (dB)	20.83	27.28	29.68	30.35	**30.90**

Signal Reconstruction Quality Factor $\mathrm{RQF}=10\log_{10}\left(\frac{\sum_n|x[n]|^2}{\sum_n|x[n]-\hat{x}[n]|^2}\right)$, of the recursive synchrosqueezed STFT computed for $k=5$, $L=7$ at $\mathrm{SNR}=45$ dB. Line (a), computed for $M=300$, Line (b) and Line (c), computed for $n_0=28$ and $M=300$.

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