Estimation of Instantaneous Frequency and Amplitude of Multi-Component Signals using Sparse Modeling of Signal Innovation

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Introduction

We consider an observed multi-component signal (MCS) made of K superimposed components:

$$x(n) = \sum_{k=0}^{K-1} x_k(n) \qquad \text{, with} \quad x_k(n) = \alpha_k(n)\, \mathbf{e}^{\frac{2\pi j \phi_k(n)}{M}}, \tag{1}$$

- with $j^2 = -1$, time instant $n \in [\![0, N-1]\!]$ and M the number of considered frequency bins.
- $\alpha_k(n)$ the Instantaneous Amplitude (IA) and $\phi_k(n)$ the instantaneous phase.
- $\phi'_k(n)$ the Instantaneous Frequency (IF) defined as the discretized derivative of ϕ_k with respect to time.

Problem statement

Goals:

- Disentengling the signal components (modes)
- IF and IA parameters estimation

Proposed approach: Time-frequency Analysis

- Efficient and intuitive framework based on the Short-Time Fourier Transform (STFT) which represents the signal in a time-frequency plane.
- Allows to observe the IF trajectory of each mode as a ridge.

Challenges

- Large variety of real-world signals (amplitude, modulation rate,. . .).
- Presence of spurious noise.
- Overlapping components.

Plan

Observation

The STFT of x, using an analysis window θ:

$$F_x^\theta(n,m) = \sum_{l=-\infty}^{+\infty} x(l)\theta(n-l)^* \, \mathbf{e}^{-j\frac{2\pi lm}{M}} \tag{2}$$

with $m \in \{0, 1, \dots, M-1\}$ and z^* being the complex conjugate of z.

Approximation

Let $\boldsymbol{S}$ be an $\mathbb{R}^{M\times N}$ matrix representing the spectrogram of x:

$$[\boldsymbol{S}]_{n,m} = |F_x^\theta(n,m)|^2 \approx \sum_{k=0}^{K-1} |x_k(n)|^2 \left| F_\theta\left(m - \phi_k'(n)\right)\right|^2 \tag{3}$$

where $F_\theta(m) = \frac{1}{M}\sum_{l\in\mathbb{Z}} \theta(l)\, \mathbf{e}^{-j\frac{2\pi lm}{M}}$.

The n-th spectrogram column is denoted as follows:

$$[\boldsymbol{S}]_{n,:} = \boldsymbol{s}_n = [s_{n,0}, \dots, s_{n,M-1}]^\top \in \mathbb{R}_+^M \tag{4}$$

Proposed finite rate of innovation (FRI) model

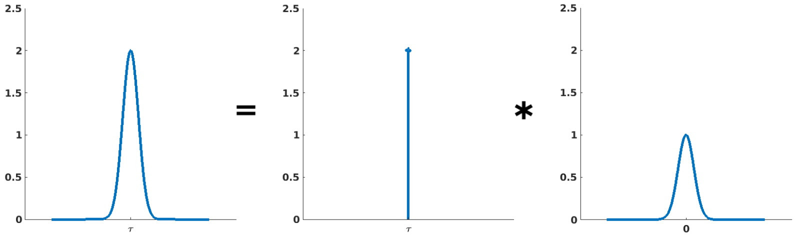
$$s_{n,m} := \sum_{k=0}^{K-1} (\alpha_k(n))^2 g(m - \phi'_k(n)) \quad (5)$$

with $g(m) = |F_\theta(m)|^2$.

Eq. (5) can be viewed as a field of Dirac Pulses (DPs) expressed as:

$$f_n(m) = \sum_{k=0}^{K-1} (\alpha_k(n))^2 \delta\left(\frac{m - \phi'_k(n)}{M}\right) \quad (6)$$

where the Dirac distribution is convolved with a known kernel g.



Model properties - [1]Vetterli et al. Sampling signals with finite rate of innovation. 2002

s_n can be reworded using the inverse DFT of $g(m - \phi'_k(n))$ as [1]:

$$
\begin{aligned}
s_{n,m} &= \sum_{k=0}^{K-1} (\alpha_k(n))^2 \sum_{\lambda=-\infty}^{\infty} F_g(\lambda)\, e^{j\frac{2\pi\lambda(m-\phi'_k(n))}{M}} \\
&= \sum_{\lambda=-\infty}^{\infty} F_g(\lambda) \underbrace{\sum_{k=0}^{K-1} (\alpha_k(n))^2\, e^{-j\frac{2\pi\lambda\phi'_k(n)}{M}}}_{F_{f_n}(\lambda)} e^{j\frac{2\pi\lambda m}{M}} \\
&\approx \sum_{\lambda=-M_0}^{M_0} F_g(\lambda) F_{f_n}(\lambda)\, e^{j\frac{2\pi\lambda m}{M}}.
\end{aligned}
\tag{7}
$$

Matrix-wise system (after bandlimited approximation)

$$
s_n = V D_g \mathcal{F}_n \Leftrightarrow \mathcal{F}_n = D_g^{-1} V^{\dagger} s_n \tag{8}
$$

- $\mathcal{F}_n = [F_{f_n}(-M_0), F_{f_n}(-M_0+1), \cdots, F_{f_n}(M_0)]^T$.
- $[V]_{m,\lambda} = e^{j\frac{2\pi m\lambda}{M}}$ is a Vandermonde Fourier matrix of size $M \times (2M_0 + 1)$.
- $V^{\dagger}$ is a pseudo-inverse of V (eg. $(V^T V)^{-1} V^T$).
- D_g is diagonal and gathers the Fourier coefficients of $F_g(\lambda)$ in $[-M_0, M_0]$.

Pulses location estimation using the Prony method

Principle: estimating the filter h that annihilates F_{f_n}:

$$(F_{f_n} \star \boldsymbol{h})(l) = \sum_{k=0}^{K-1} (\alpha_k(n))^2 \, \mathbf{e}^{-j\frac{2\pi l \phi'_k(n)}{M}} H\left(\mathbf{e}^{-j\frac{2\pi \phi'_k(n)}{M}}\right) = 0 \qquad (9)$$

with $H(z) = \sum_{i\in\mathbb{Z}} h(i) z^{-i}$ whose the roots are $\mathbf{e}^{-j\frac{2\pi\phi'_k(n)}{M}}$.

Yule-Walker system [2]

Assuming $h(0) = 1$, $\boldsymbol{h}$ can be estimated solving the following linear system which has a unique solution if $M_0 \geq 2K$:

$$\underbrace{\begin{pmatrix} F_{f_n}(0) & \cdots & F_{f_n}(-K+1) \\ F_{f_n}(1) & \cdots & F_{f_n}(-K+2) \\ \vdots & \ddots & \vdots \\ F_{f_n}(K-1) & \cdots & F_{f_n}(0) \end{pmatrix}}_{\boldsymbol{A}} \begin{pmatrix} h(1) \\ h(2) \\ \vdots \\ h(K) \end{pmatrix} = - \begin{pmatrix} F_{f_n}(1) \\ F_{f_n}(2) \\ \vdots \\ F_{f_n}(K) \end{pmatrix} \qquad (10)$$

[2] T. Blu and P. L. Dragotti and M. Vetterli and P. Marziliano and L. Coulot. Sparse sampling of signal innovations, 2008.

IF estimation

Classical approach

FRI Prony method

- Accurate but sensitive to noise
- Not applicable for real-world signals

Total Least-Squares (TLS) approach

Minimizes $\|\boldsymbol{A}h\|^2$ under the constraint $\|h\|_2 = 1$.

- Inverting the Vandermonde matrix becomes an ill-posed problem.
- h does not annihilate $F_{f_{\boldsymbol{n}}}$ anymore.
- Can be solved using the Singular Values Decomposition (SVD) of $\boldsymbol{A}$, $\boldsymbol{h}$ being the eigenvector associated with the smallest eigenvalue.

IA estimation

Classical FRI method

$$\begin{pmatrix} W_{0,0} & \cdots & W_{0,K-1} \\ W_{1,0} & \cdots & W_{1,K-1} \\ \vdots & \ddots & \vdots \\ W_{K-1,0} & \cdots & W_{K-1,K-1} \end{pmatrix} \begin{pmatrix} (\alpha_0(n))^2 \\ (\alpha_1(n))^2 \\ \vdots \\ (\alpha_{K-1}(n))^2 \end{pmatrix} = \begin{pmatrix} F_{f_n}(0) \\ F_{f_n}(1) \\ \vdots \\ F_{f_n}(K-1) \end{pmatrix} \quad (11)$$

where $W_{l,k} = e^{-j\frac{2\pi l \phi'_k(n)}{M}}$. Least Squares estimation:

$$\hat{\alpha}_k^{(LS)}(n) = \underset{\alpha_k(n)}{\mathrm{argmin}} \sum_{|\lambda| \leq M_0} \left| F_{f_n}(\lambda) - \sum_{k=0}^{K-1} (\alpha_k(n))^2 \, \mathbf{e}^{-j\frac{2\pi\lambda\phi'_k(n)}{M}} \right|^2 . \quad (12)$$

Proposed method

$$\hat{\alpha}_k(n) = \left| \frac{F_x^{\theta}(n, m_k)}{F_{\theta}(m_k - \phi'_k(n))} \right| \quad (13)$$

with $m_k \in \{0, 1, \cdots, M-1\}$ being the nearest integer frequency bin (on the grid) from $\phi'_k(n)$

STFT/SST Recursive implementation - [3] Fourer, Auger, Flandrin. ICASSP 2016

- allows real-time and filter-bank-based signal processing applications
- use of a specific causal analysis window related to an IIR filter

$$\theta_p(n) = \frac{n^{p-1}}{L^p(p-1)!}\,\mathbf{e}^{-n/L}U(n) \tag{14}$$

where p is the filter order, L the spread of the analysis window and $U(n)$ the Heaviside function.

$$\text{As a results:}\quad g(m) = |F_{\theta_p}(m)|^2 = \left(1 + \left(\frac{2\pi mL}{M}\right)^2\right)^{-p} \tag{15}$$

with $F_{\theta_p}(m) = (1 + j\frac{2\pi mL}{M})^{-p}$.
Thus, $F_x^{\theta_p}(n,m) = y_p(n,m)\,\mathbf{e}^{j\frac{2\pi nm}{M}}$ can be computed recursively using:

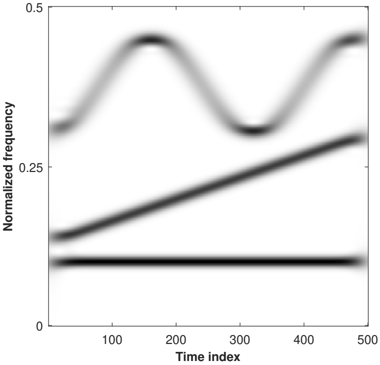
$$y_p(n,m) = \sum_{i=0}^{p-1} b_i\, x(n-i) - \sum_{i=1}^{p} a_i\, y_p(n-i,m) \tag{16}$$

a_i and b_i being the filter coefficients resulting from the z-transform of the Infinite Impulse Response (IIR) filter $\theta_p(n)\,\mathbf{e}^{j\frac{2\pi nm}{M}}$.

Results

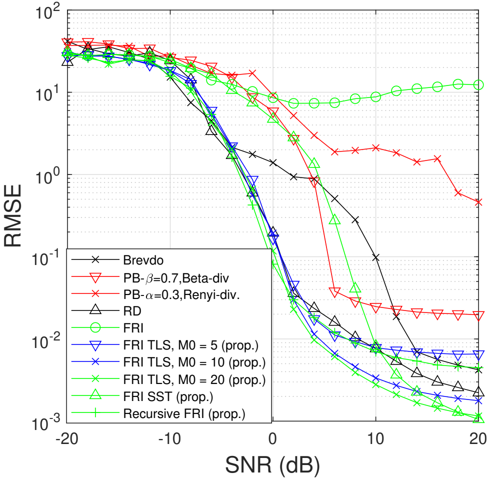
Experiments

- Synthetic (3-components signal) and real data.
- Comparison with various approaches [4][5][6][7]
- IF assessed with RMSE
  IA assessed with RMAE.
- White Gaussian noise with various Signal-to-noise ratio (SNR).



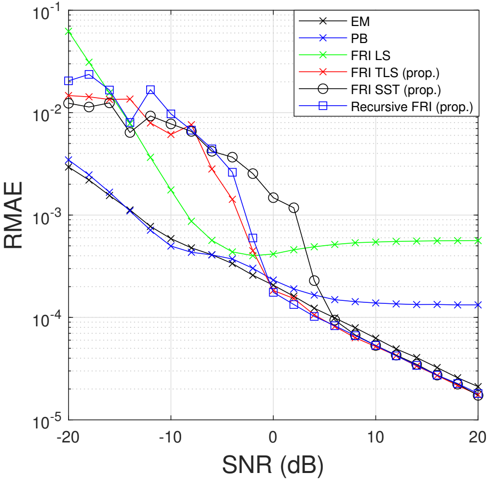
- [4] E. Brevdo and N. S. Fuckar and G. Thakur and H. T. Wu. The synchrosqueezing algorithm: a robust analysis tool for signals with time-varying spectrum, 2011.
- [5] Q. Legros, D. Fourer. Pseudo-Bayesian Approach for Robust Mode Detection and Extraction Based on the STFT, 2023.
- [6] N. Laurent and S. Meignen. A Novel Ridge Detector for Nonstationary MCS: Development and Application to Robust Mode Retrieval, 2021.
- [7] Q. Legros and D. Fourer and S. Meignen and M. A. Colominas. Instantaneous Frequency and Amplitude Estimation in Multicomponent Signals Using an EM-Based Algorithm, 2024.

Comparative Results - IF estimation



Relative Mean Squared Error: $\text{RMSE}(\phi', \hat{\phi}') = \sum_{k=0}^{K-1} \sum_{n=0}^{N-1} \frac{|\phi'_k(n) - \hat{\phi}'_k(n)|^2}{M^2}$

Comparative Results - IA estimation



Relative Mean Absolute Error: $\mathrm{RMAE}(\alpha, \hat{\alpha}) = \frac{1}{NK} \sum_{k=0}^{K-1} \sum_{n=0}^{N-1} |\alpha_k(n) - \hat{\alpha}_k(n)|$

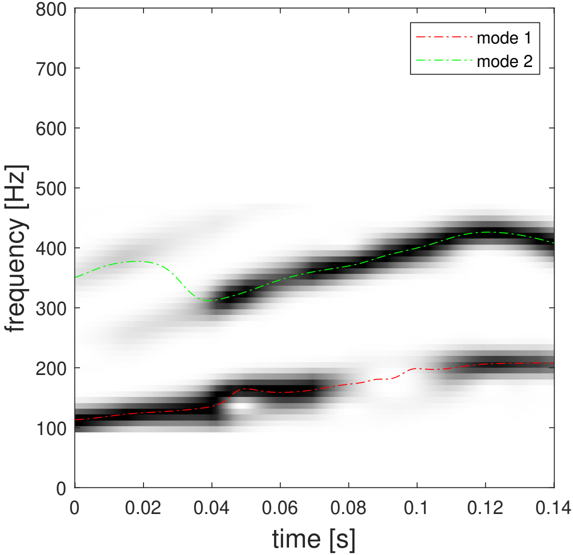
Results - Computation times

- Matlab R2023b
- Intel(R) Core(TM) i7-12700H @ 2.30 GHz

Table : Computation time expressed in seconds for synthetic data analysis, averaged over 50 realizations.

M	500	1000	2000
Brevdo	0.13	0.14	0.18
EM	6.93	29.36	116.42
PB	0.23	0.36	0.61
RD	0.41	0.54	1.20
FRI TLS (proposed)	**0.12**	**0.12**	0.2
Recursive FRI (proposed)	**0.12**	0.13	**0.13**

Real-world speech signal analysis



Conclusions and perspectives

Conclusions

- A novel observation model for IF and IA estimation of modes within a multicomponent signal in the presence of noise.
- Time-independent estimation allows recursive implementation.
- Computationally light and robust to noise.

Future works

- Overlapping components: constraining the problem
- Close ridges, interference : spatial regularization.

Thanks for your attention !

Codes available on GitHub
`https://github.com/QuentinLEGROS/EUSIPCO2024/`

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